ON VOLTERRA INTEGRAL EQUATIONS WITH
NONNEGATIVE INTEGRABLE RESOLVENTS

by

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I. INTRODUCTION

Given the linear Volterra integral equation

$$(1) \qquad X(t) = F(t) - \int_0^t a(t-s)X(s)ds, \qquad (t \geq 0)$$

it is well known that the solution has the form

$$(2) \qquad X(t) = F(t) - \int_0^t k(t-s)F(s)ds$$

where the resolvent kernel $k(t)$ is the solution of the linear equation

$$(R) \qquad k(t) = a(t) - \int_0^t a(t-s)k(s)ds.$$

In section II below we give sufficient conditions on the function $a(t)$ in order that $k(t)$ be both nonnegative and of class $L^1(0,\infty)$. We give simple formulas for calculating $\int_0^\infty k(t)dt$. These results give detailed information concerning the solution (2) of the linear equation (1) when the function $F(t)$ is known. It is also shown that if $a(t)$ is nonconstant, locally integrable and completely monic on $0 < t < \infty$, then $a(t)$ satisfies our sufficient conditions. This shows that our criteria are satisfied by a large class of interesting and important kernels $a(t)$.

The results of section II also give information concerning the behavior of certain nonlinear equations. Consider the equation

$$x(t) = F(t) - \int_0^t a(t-s)g(x(s))ds, \tag{3}$$

where $g(x) = x+o(x)$ as $|x| \to 0$. According to the theory developed in [8] if F is "small", the solution $k(t)$ of equation (R) is $L^1(0,\infty)$ and the solution $X(t)$ of equation (1) tends to zero as $t \to \infty$, then the solution $x(t)$ of (3) tends to zero. If $a \in L^1(0,\infty)$, then a well known result of Paley-Weiner provides necessary and sufficient conditions in order that k be of class $L^1(0,\infty)$, c.f. [12, p. 60]. Nohel [10] has pointed out that in order to widen the range of application of the results in [8] it would be of interest to develope other general criteria which guarantee $k(t) \in L^1(0,\infty)$. The results in section II are a partial solution to this problem.

In sections III, IV and V below the results of section II are applied to the study of the behavior of solutions of nonlinear equations of the form

$$x(t) = \int_0^t a(t-s)G(x(s),s)ds \qquad (t \geq 0) \tag{N}$$

when $G(x,t)$ is nonincreasing in x (for each fixed t) and $a(t)$ is such that $k(t)$ is both nonnegative and of class $L^1(0,\infty)$. Section III contains preliminary results which are needed in the sequel. Section IV contains results concerning the behavior of $x(t)$ when $G(x,t)$ is nondecreasing in t. In section V we show that if G is 2Π-periodic in its second variable and satisfies some other mild conditions, then the solution $x(t)$ of equation (N) tends to a certain periodic function $\varphi(t)$ as $t \to \infty$.

The results in section II are related to certain results of Friedman [2, 3]. Under the addition assumption that $a \in C^2(0,\infty)$ or that $a(0^+) < \infty$, Theorem 2 can be proved using Friedman's techniques. Corollary 2 was proved by Friedman using different techniques, c.f.[2, Theorem 8]. The results in sections III and IV are related to results of Mann and Wolf [6], Mann and Roberts [5], Padmavalley [11] and Friedman [2]. They generalize all of the main results in [5] and [11]. Friedman [2, p. 391] has suggested a different method of obtaining the results in section IV under slightly stronger smoothness assumptions on the function G.

Theorem 10 of section V generalizes certain results of Levinson [4, Theorem 2] and Friedman [3, Theorem 5]. Under very general assumptions on $a(t)$ and G Friedman shows that the solution $x(t)$ of equation (N) satisfies the condition

$$\lim\{x(t+2\Pi) - x(t)\} = 0 \qquad (t \to \infty). \tag{4}$$

He askes whether or not condition (4) can be replaced by the stronger assertion that $x(t)$ tends to a 2Π-periodic function as $t \to \infty$. The results in section V show that there does exist a 2Π-periodic function φ such that

$$\lim\{x(t) - \varphi(t)\} = 0 \qquad (t \to \infty).$$

The basic results on local existence, uniqueness and continuation of solutions of nonlinear Volterra integral equations will be needed in the sequel. Local existence and uniqueness theorems are in most texts. The necessary results on continuation as well as very general existence and uniqueness theorems may be found in [9].

II. THE RESOLVENT KERNEL

The following result of Friedman [2, p. 387] concerning the equation

$$x(t) = f(t) - \int_0^t h(t-s)g(x(s),s)ds \tag{5}$$

will be needed. This result is a generalization of Lemmas 1, 1a and 1b of [11]. Since the proof is not completely clear from Friedman's remarks, a proof is given here.

THEOREM 1. *Suppose f is positive and continuous on the interval*

$0 \leq t < \infty$. Let h be positive, continuous and locally integrable on $0 < t < \infty$. Suppose $g(x,t)$ is measurable in (x,t) for $-\infty < x < \infty$, $0 \leq t < \infty$, g is continuous in x for each fixed t, $xg(x,t) \geq 0$ for all (x,t) and the functions f, h and g are sufficiently smooth to guarantee the uniqueness of the solution of (5). If

$$f(T)/f(t) \leq h(T-s)/h(t-s)$$

whenever $0 \leq s \leq T < t$, then the solution $x(t)$ of (5) exists for all $t \geq 0$ and satisfies $0 \leq x(t) \leq f(t)$.

Proof. Define $g^*(x,t) = g(x,t)$ if $x \geq 0$ and $g^*(x,t) = 0$ if $x \leq 0$. Let $x^*(t)$ be a solution of (5) when g is replaced by g^*. Then for as long as x^* exists it must be nonnegative. If this were not true, then the set

$$A = \{t \geq 0;\ x^*(s) \text{ exists on } [0,t] \text{ and } x^*(t) < 0\}$$

is a nonempty open set. Let (T,T_o) be a maximal open interval contained in A. Then $x^*(T) = 0$ and for $T < t < T_o$

$$\begin{aligned} 0 > x^*(t) &= f(t) - \int_0^T h(t-s)g^*(x^*(s),s)ds \\ &= \{f(t)/f(T)\}\{f(T) - \int_0^T f(T)h(t-s)g^*(x^*(s),s)/f(t)ds\} \\ &\geq \{f(t)/f(T)\}\{f(T) - \int_0^T h(T-s)g^*(x^*(s),s)ds\} \\ &= \{f(t)/f(T)\}x^*(T) = 0. \end{aligned}$$

This contradiction shows that A is empty. Since x^* is non-negative, $0 \leqq x^*(t) \leqq f(t)$ for as long as x^* exists. Since $f(t)$ is bounded on each finite interval, $x^*(t)$ can be continued as a solution of the integral equations for all $t \geqq 0$. The definition of g^* implies that $x^*(t) = x(t)$. This completes the proof of Theorem 1.

Concerning the function $a(t)$ it is assumed that

(A1) $a \in L^1(0,1)$,

(A2) a is positive, continuous and nonincreasing on the interval $0 < t < \infty$, and

(A3) for any $T > 0$ the function $a(t)/a(t+T)$ is a nonincreasing function of t on the interval $0 < t < \infty$ (compare [5, p. 432]).

<u>THEOREM</u> 2. <u>If</u> $a(t)$ <u>satisfies</u> (A1-3), <u>then equation</u> (R) <u>has a unique, locally integrable solution</u> k. <u>The function</u> k <u>exists and is continuous on</u> $0 < t < \infty$. <u>If</u> $a(0^+) < \infty$, <u>then</u> k <u>is also defined and continuous at</u> $t = 0$. <u>Moreover</u>

(i) $0 \leqq k(t) \leqq a(t)$ <u>on</u> $0 < t < \infty$,

(ii) $\int_0^\infty k(t)dt \leqq 1$, <u>and</u>

(iii) $k(t) \not\equiv 0$ <u>on any interval of the form</u> $(0,T)$, $T > 0$.

<u>Proof</u>. Let ε_n be a positive sequence decreasing to zero. Define $a_n(t) = a(t+\varepsilon_n)$ and let k_n be the unique solution of

$$(6) \qquad k_n(t) = a_n(t) - \int_0^t a(t-s)k_n(s)ds. \qquad (t \geq 0)$$

If follows from Theorem 1 above that $0 \leq k_n(t) \leq a_n(t) \leq a(t)$ for $0 < t < \infty$.

Since $k_n(t) \geq 0$ and $a(t)$ is nonincreasing

$$\int_0^t |k_n(s)|ds = \int_0^t k_n(t-s)ds \leq \int_0^t \{k_n(t-s)a(s)\}/a(t)ds$$
$$= \{a_n(t) - k_n(t)\}/a(t) \leq 1.$$

Letting $t \to \infty$ we see that $\int_0^\infty |k_n(s)|ds \leq 1.$

Fix any $T > 0$ and define $K_n(t) = k_n(t)$ on $0 \leq t \leq T$ and $K_n(t) = 0$ elsewhere. Then

$$\int_{-\infty}^\infty |K_n(s)|ds \leq 1. \qquad (n = 1, 2, 3, \ldots)$$

For any h $(0 < h < 1)$ we have

$$k_n(t+h) - k_n(t) = a_n(t+h) - a_n(t) - \int_t^{h+t} a(t+h-s)k_n(s)ds$$
$$- \int_0^t \{a(s+h) - a(s)\}k_n(t-s)ds.$$

Since $0 \leq k_n(t) \leq a(t)$ it follows that

$$\int_h^T\int_t^{t+h} a(t+h-s)k_n(s)dsdt \leq \int_o^T\int_o^h a(h-s)k_n(t+s)dsdt$$

$$= \int_o^h(\int_o^T k_n(t+s)dt)a(h-s)ds \leq \int_o^h 1\cdot a(h-s)ds = \int_o^h a(s)ds.$$

Similiarly it follows that

$$\int_h^T\int_o^t |a(s+h) - a(s)|k_n(t-s)dsdt \leq \int_o^T\{\int_s^T k_n(t-s)dt\}|a(s+h) - a(s)|ds$$

$$\leq \int_o^T 1\cdot|a(s+h) - a(s)|ds.$$

Therefore when $\varepsilon_n < 1$

$$\int_h^T|k_n(t+h) - k_n(t)|dt \leq \int_o^h a(s)ds + 2\int_o^T|a(s+h) - a(s)|ds.$$

Since for each n, $k_n(t) \leq a(t)$ it follows that

$$\int_{-\infty}^{\infty}|K_n(s+h) - K_n(s)|ds \leq \int_o^h k_n(t)dt + \int_T^{T+h} k_n(t)dz + \int_h^T|k_n(t+h) - k_n(t)|dt \to 0$$

as $h \to 0^+$ uniformly for all n. A similiar result holds when $h \to 0^-$. This shows that the sequence $\{K_n\}$ has compact closure in the space $L^1(-\infty,\infty)$, c. f. [1, pp. 298-299].

There is a function $k_o \in L^1(0,T)$ and a subsequence (which we again index by n) such that $k_n \to k_o$ in $L^1(0,T)$. By possibly taking a further subsequence we may assume that $k_n(t) \to k_o(t)$ a. e. Repeating the above argument for the successive intervals $(0,nT)$,

$n = 2, 3, 4, \ldots$ and using the diagonal subsequence we may define $k_o(t)$ on all of $(0,\infty)$.

Since $k_n(t) \to k_o(t)$ a. e. it follows that $0 \leq k_o(t) \leq a(t)$ a. e. on $(0,\infty)$. Clearly $\int_o^\infty k_o(t)dt \leq 1$. Since a. e. $|k_n(s) - k_o(s)| \leq 2a(s)$, it follows from dominated convergence that a. e.

$$\int_o^t a(t-s)\{k_n(s) - k_o(s)\}ds \to 0$$

as $n \to \infty$. Therefore k_o solves equation (R) a. e. For $0 < t < \infty$ define

$$k(t) = a(t) - \int_o^t a(t-s)k_o(s)ds.$$

Then $k(t)$ is continuous on $0 < t < \infty$ and $k(t) = k_o(t)$ a. e. Thus k solves (R) and satisfies (i) and (ii). Since $a(0^+) = k(0^+)$ one can define $k(0) = a(0^+)$ if $a(0^+) < \infty$.

In order to prove uniqueness suppose $k_1 \in L^1(0,T)$, is continuous on $0 < t < T$ and solves (R). Fix an integer n so large that if $h = T/n$, then $\int_o^h a(s)ds < 1$. If $k_1(t) \not\equiv k(t)$ on $[0,h]$ then

$$\int_o^h |k_1(s)-k(s)|ds = \int_o^h \left|\int_o^t a(s)\{k_1(t-s)-k(t-s)\}ds\right| dt$$

$$\leq \int_o^h \{\int_s^h |k_1(t-s)-k(t-s)|dt\}a(s)ds$$

$$\leq \int_o^h \{\int_o^h |k_1(t)-k(t)|dt\}a(s)ds < \int_o^h |k_1(t)-k(s)|ds.$$

Therefore $k_1(t) = k(t)$ on $(0,h]$. Since

$$k(t+h) = \{a(t+h) - \int_0^h a(t+h-s)k(s)ds\} - \int_0^t a(t-s)k(s+h)ds$$

The same argument shows that

$$\int_0^h |k_1(t+h) - k(t+h)|\,dt = 0,$$

that is $k_1(t) = k(t)$ for $h \leqq t \leqq 2h$. By induction $k(t) = k_1(t)$ on $(0,T]$. This proves the uniqueness of k.

To prove (iii) suppose $k(t) = 0$ for all t on an interval $0 < t < T$. Then it follows from equation (R) that $a(t) = 0$ a. e. on $(0,T)$. This contradicts the assumption $a(t) > 0$ so that (iii) follows. This completes the proof of Theorem 2.

<u>Corollary</u> 1. <u>Suppose</u> $a(t)$ <u>satisfies</u> (A1-3).

(i) <u>If</u> $a(t) \notin L^1(0,\infty)$, <u>then</u> $\int_0^\infty k(t)dt = 1$.

(ii) <u>If</u> $\int_0^\infty a(t)dt = A < \infty$, <u>then</u>

$$\int_0^\infty k(t)dt = A(1+A)^{-1} < 1.$$

<u>Proof</u>. Suppose $a(t) \notin L^1(0,\infty)$. The solution $x(t)$ of the equation

$$x(t) = 1 - \int_0^t a(t-s)x(s)ds$$

is $x(t) = 1 - \int_0^t k(s)ds$. By Theorem 2 above $x(t)$ is nonnegative and nonincreasing.

Suppose that $\lim x(t) = x(\infty) > 0$ $(t \to \infty)$. Then

$$x(t) \leqq 1 - \int_0^t a(t-s)x(\infty)ds \to -\infty$$

as $t \to \infty$. This contradicts $x(t) \geqq 0$ and proves part (i).

Now suppose $a(t) \in L^1(0,\infty)$. If $*$ denotes the Laplace transformation, then it follows from equation (R) that

$$k^*(w) = a^*(w) - a^*(w)k^*(w),$$

when $\operatorname{Re} w \geqq 0$. Since $a^*(0) = A$, part (ii) follows immediately. This proves Corollary 1.

Definition: <u>A function</u> $b(t)$ <u>is completely monic on</u> $0 < t < \infty$ <u>if and only if</u> $b \in C^\infty(0,\infty)$ <u>and</u>

$$(-1)^j b^{(j)}(t) \geqq 0. \quad (j = 0, 1, 2, \ldots, 0 < t < \infty)$$

It is known, c.f. [13, p. 161] that a function $a(t)$ is completely monic on $(0,\infty)$ if and only if there exists a nondecreasing function γ such that

$$a(t) = \int_0^\infty \exp(-st)\,d\gamma(s), \qquad (0 < t < \infty) \tag{7}$$

where the integral is absolutely convergent for each t in $(0,\infty)$.

LEMMA 1. *If* $a(t)$ *is completely monic on* $0 < t < \infty$, *then* $a(t) \equiv 0$ *or* $a(t) > 0$ *for all* $t \in (0,\infty)$.

Proof. If $a(t_o) = 0$ for some $t_o > 0$, then since $a'(t) \leqq 0$ we see that $a(t) = 0$ for all $t \geqq t_o$. Since line (7) implies that $a(t)$ is the restriction to the positive reals of an analytic function, Lemma 1 is proved.

LEMMA 2. *If* $a(t)$ *is completely monic on* $0 < t < \infty$ *and* $a(t) \not\equiv 0$ *then* $a(t)$ *satisfies* (A3).

Proof. Assumption (A3) is equivalent to the convexity of the function $\log a(t)$. This in turn is equivalent to the condition that

$$y(t) = a(t)a''(t) - (a'(t))^2 \geqq 0. \qquad (0 < t < \infty)$$

Using (7) we see that

$$y(t) = \int_0^\infty \exp(-wt)d\gamma(w)\int_0^\infty z^2\exp(-zt)d\gamma(z) - \int_0^\infty w\exp(-wt)d\gamma(w)\int_0^\infty z\exp(-zt)d\gamma(z),$$

$$= \int_0^\infty\int_0^\infty z(z-w)\exp(-t(z+w))d\gamma(z)d\gamma(w).$$

Using the absolute convergence of the integrals it follows that

$$\int_0^\infty\int_w^\infty z(z-w)\exp(-t(z+w))d\gamma(z)d\gamma(w) = \int_0^\infty\int_0^z z(z-w)\exp(-t(z+w))d\gamma(w)d\gamma(z)$$

$$= \int_0^\infty\int_0^w w(w-z)\exp(-t(w+z))d\gamma(z)d\gamma(w).$$

Therefore

$$y(t) = \int_0^\infty\int_0^w(\ldots) + \int_0^\infty\int_w^\infty(\ldots) = \int_0^\infty\int_0^w(z-w)^2\exp(-t(z+w))d\gamma(z)d\gamma(w) > 0.$$

This proves Lemma 2.

Corollary 2. *If $a(t)$ is nonconstant, locally integrable and completely monic on the interval $0 < t < \infty$, then the solution $k(t)$ of equation (R) satisfies all of the conclusions of Theorem 2 and Corollary 1 above.*

Proof. This follows immediately from Lemmas 1 and 2 above.

Now consider the sequence of function $\{k_n\}$ defined by line (6) of the proof of Theorem 2. It was shown that there exists a subsequence such that for any $T > 0$ this subsequence converges in the sense $L^1(0,T)$ to the solution k. Since the solution k of equation

(R) is unique, it is easily shown that the entire sequence $k_n \to k$ in the L^1 sense on each interval $[0,T]$. If $a(t)$ is completely monic, more can be said about the convergence $k_n \to k$.

THEOREM 3. *Let* $a(t)$ *be nonconstant, locally integrable and completely monic on the interval* $0 < t < \infty$. *Suppose* ε_n, a_n *and* k_n *are as defined in the proof of Theorem 2 above. Then*

(i) *for each* $t > 0$ *the sequence* $\{k_n(t)\}$ *is nondecreasing with* $k_n(t) \to k(t)$, *and*

(ii) *for each pair of constants* t_o *and* t_1 *with* $0 < t_o < t_1$ *the sequence* $k_n(t) \to k(t)$ *uniformly for* $t \in [t_o, t_1]$.

Proof. Fix m and n with $m > n$. Define

$$z(t) = k_m(t) - k_n(t) \quad \text{and} \quad f(t) = a_m(t) - a_n(t).$$

Then z satisfies

$$z(t) = f(t) - \int_o^t a(t-s)z(s)ds.$$

In order to apply Theorem 1 above we must show that when $0 \leq \tau \leq T < t$,

$$y = f(T)a(t-\tau) - f(t)a(T-\tau) \leq 0.$$

Using (7) we see that

$$f(t) = \int_0^\infty \exp(-s(t+\varepsilon_m))\{1-\exp(-s\delta)\}d\gamma(s)$$

where $\delta = \varepsilon_n - \varepsilon_m > 0$. Therefore

$$y = \int_0^\infty \int_0^\infty \{1-\exp(-s\delta)\}F(s,r)d\gamma(s)d\gamma(r),$$

where

$$F(s,r) = \exp(-rt+r\tau-sT-s\varepsilon_m) - \exp(-rT+r\tau-st-s\varepsilon_m).$$

Split y into the two integrals

$$y = \int_0^\infty \int_0^r (\ldots) + \int_0^\infty \int_r^\infty (\ldots) = I_1 + I_2.$$

If we reverse the order of integration in I_2, interchange the roles of r and s, and then add I_1 and I_2 it follows that

$$y = \int_0^\infty \int_0^r A(BC-DE)d\gamma(s)d\gamma(r),$$

where

$$A = \exp(-rt-sT)-\exp(-rT-st),$$

$$B = \exp(r\tau-s\varepsilon_m), \quad C = 1-\exp(-s\delta),$$

$$D = \exp(s\tau-r\varepsilon_m), \quad E = 1-\exp(-r\delta).$$

Since τ, δ and $t - T > 0$ and $0 < s < r$ it is easily shown that $-A$, $B-D$ and $C-E$ are positive. Thus $y \leqq 0$, $z(t) \geqq 0$ for all $t > 0$. Therefore whenever $m > n$

$$(8) \qquad 0 \leqq k_m(t) - k_n(t) \leqq a(t+\varepsilon_m) - a(t+\varepsilon_n)$$

uniformly for all $t > 0$.

Since the sequence $\{k_n(t)\}$ is nondecreasing and is bounded above by $a(t)$, it converges to a limit $k_1(t)$. Line (8) implies that $\{k_n(t)\}$ is a Cauchy sequence uniformly on each compact subset of $0 < t < \infty$. Therefore $k_1(t)$ is continuous and must equal $k(t)$ for all $t > 0$. This proves part (i).

Letting $m \to \infty$ in (8) it follows that

$$0 \leqq k(t) - k_n(t) \leqq a(t) - a(t+\varepsilon_n),$$

uniformly for all $t > 0$. This proves part (ii) and completes the proof of Theorem 3.

Note that in case $a(t)$ is completely monic, $k(t)$ is also completely monic. By Theorem 8 of [2] the solution $x(t)$ of the equation

$$x(t) = 1 - \int_0^t a(t-s)x(s)ds$$

is completely monic. Since $x(t) = 1-\int_0^t k(s)ds$, $x^{(j)}(t) = -k^{(j-1)}(t)$ for $j = 1, 2, 3,\ldots$ and the remark follows.

III. PRELIMINARY RESULTS

This section contains generalizations of some of the results in [11].

LEMMA 3. *Suppose* $a(t)$ *satisfies* (A1-3). *For some fixed* $T > 0$ *let* φ *and* θ *be nonnegative with* $\theta \in L^\infty(0,T)$ *and* $\varphi \in L^\infty(0,T)$. *Then the solution* z *of*

$$z(t) = \int_0^t a(t-s)\theta(s)ds - \int_0^t a(t-s)\varphi(s)z(s)ds \tag{9}$$

exists on the interval $0 \leq t \leq T$ *and is nonnegative there.*

Proof. Let $M = 1 + \text{ess. sup. } \varphi(t)$, $0 \leq t \leq T$. Then (9) is equivalent to the equation

$$z(t) = \int_0^t Ma(t-s)\{\theta(s)/M\}ds - \int_0^t Ma(t-s)\{\varphi(s)/M\}z(s)ds.$$

Since $Ma(t)$ also satisfies (A1-3) it is no loss of generality to assume that $0 \leq \varphi(s) \leq 1$ on $[0,T]$.

Equation (9) may be written in the form

$$z(t) = f(t) - \int_0^t a(t-s)z(s)ds,$$

where

$$f(t) = \int_0^t a(t-s)\{\theta(s) + (1-\varphi(s))z(s)\}ds.$$

If k is the solution of (R), then it follows that

$$z(t) = f(t) - \int_0^t k(t-s)f(s)ds,$$

or

$$(10) \qquad z(t) = \int_0^t k(t-s)\theta(s)ds + \int_0^t k(t-s)(1-\varphi(s))z(s)ds.$$

Since the reasoning is reversible, equations (9) and (10) are equivalent. Since the coefficients in equation (10) are nonnegative, $z(t)$ must be nonnegative for as long as it exists. The nonnegativity of z and (9) imply

$$0 \leqq z(t) \leqq \int_0^t a(t-s)\theta(s)ds$$

for as long as z exists. Thus $z(t)$ exists on $[0,T]$ and Lemma 3 is proved.

Corollary 3. *If in Lemma 3 one has* $\theta(t) > 0$ a. e., *then* $z(t) > 0$ *for* $0 < t \leqq T$.

Proof. Line (10) implies that $z(t) \geq \int_0^t k(t-s)\theta(s)ds$. Using Theorem 2, part (iii) it follows that

$$\int_0^t k(t-s)\theta(s)ds > 0$$

when $0 < t \leq T$. This proves Corollary 3.

LEMMA 4. *Suppose* a, φ, θ *and* T *satisfy the hypotheses of Lemma 3. Suppose* z *solves*

$$z(t) = \int_0^h a(t+h-s)\theta(s)ds - \int_0^t a(t-s)\varphi(s)z(s)ds, \tag{11}$$

where $0 < h < T$. *Then* z *exists and is nonnegative on* $[0,T]$.

Proof. As in the proof of Lemma 3 one may assume that $0 \leq \varphi(t) \leq 1$. Also note that (11) is equivalent to

$$z(t) = f(t) + \int_0^t k(t-s)(1-\varphi(s))z(s)ds, \tag{12}$$

where

$$f(t) = \int_0^h a(t+h-s)\theta(s)ds - \int_0^t\int_0^h k(t-s)a(s+h-u)\theta(u)duds.$$

Using (R) and some manipulation it follows that

$$\int_0^t\int_0^h k(t-s)a(s+h-u)\theta(u)duds = \int_0^h\{\int_0^t k(t-s)a(s+h-u)ds\}\theta(u)du$$

$$= \int_0^h\{\int_0^{t+h-u} k(t+h-u-s)a(s)ds - \int_{u-h}^0 k(t-s)a(s+h-u)ds\}\theta(u)du$$

$$= \int_0^h\{a(t+h-u) - k(t+h-u) - \int_{u-h}^0 k(t-s)a(s+h-u)ds\}\theta(u)du.$$

Therefore

$$f(t) = \int_0^h\{k(t+h-u) + \int_{u-h}^0 k(t-s)a(s+h-u)ds\}\theta(u)du \geqq 0.$$

Since the coefficients of equation (12) are nonnegative, $z(t) \geqq 0$. Thus (11) implies that for as long as z exists

$$0 \leqq z(t) \leqq \int_0^h a(t+h-s)\theta(s)ds.$$

Therefore z exists on the interval $[0,T]$ and Lemma 4 is proved.

LEMMA 5. *Suppose* $a(t)$ *satisfies* (A1-3). *If* k *is the solution of* (R), *then for each finite* $T > 0$

$$\int_0^T k(t)dt < 1.$$

Proof. The function $x(t) = 1-\int_0^t k(s)ds$ is the solution of

$$x(t) = 1-\int_0^t a(t-s)x(s)ds.$$

Suppose for contradiction that $x(t) = 0$ for all $t \geqq T > 0$. We may assume that T is the smallest such number. If $t_1 > t_o > T$, then

$$x(t_j) = 0 = 1 - \int_o^T a(t_j - s)x(s)ds,$$

and

$$\int_o^T \{a(t_o - s) - a(t_1 - s)\} x(s)ds = 0.$$

Since $a(t)$ is nonincreasing and $x(s) > 0$ when $0 \leqq s < T$, $a(t_o - s) = a(t_1 - s)$ when $0 \leqq s < T$. Since t_o and t_1 are arbitrary, $a(t) \equiv A$, a constant, for all $t \geqq 0$. Equation (R) can then be solved for $k(t) = A \exp(-At)$. For this $k(t)$ no such T can exist. This proves Lemma 5.

Corollary 5. _Suppose the hypotheses of Lemma 4 are satisfied. If_ $\theta(t) > 0$ a. e. _then the solution_ $z(t)$ _of_ (11) _is positive when_ $0 \leqq t \leqq T$.

Proof. It follows from line (12) of the proof of Lemma 4 that $z(t) \geqq f(t)$. In order to show that $f(t) > 0$ it is sufficient to show that $k(t) > 0$ on the interval $0 < t < \infty$.

If $k(t) = 0$ for any $t > 0$ then equation (R) has the form

$$0 = a(t) - \int_0^t a(t-s)k(s)ds.$$

Using Lemma 5 and the assumption that $a(t)$ is nonincreasing it follows that

$$a(t) = \int_0^t a(t-s)k(s)ds \geqq a(t)\int_0^t k(s)ds > a(t).$$

The contradiction shows that $k(t) > 0$ on any interval. This proves Corollary 5.

Now consider the nonlinear equation

(N) $$x(t) = \int_0^t a(t-s)G(x(s),s)ds, \qquad (t \geqq 0)$$

together with the following assumptions:

(A4) G is measurable in (x,t) for $-\infty < x < \infty$, $0 \leqq t < \infty$, continuous and nonincreasing in x for each fixed t, and bounded on each finite rectangle $x_1 \leqq x \leqq x_2$, $0 \leqq t \leqq t_1$.

(A5) There is a function $u(t)$, bounded on each finite subinterval of $[0,\infty)$, such that $G(u(t),t) = 0$ for all $t \geqq 0$.

(A6) $G(x,t)$ is locally Lipschitz continuous in x.

THEOREM 4. If $a(t)$ satisfies (A1-3) and G satisfies (A4-6), then equation (N) has a unique local solution.

The proof of Theorem 4 is well known.

THEOREM 5. *Suppose* $a(t)$ *satisfies* (A1-3) *and both* G_1 *and* G_2 *satisfy* (A4-6) *with* $G_2(x,t) \leq G_1(x,t)$ *for all* (x,t). *Let* y_i *solve* (N) *when* $G = G_i$, $i = 1, 2$. *Suppose both* y_1 *and* y_2 *exist on an interval* $[0,T]$.

(i) *Then* $y_2(t) \leq y_1(t)$ *on* $[0,T]$.

(ii) *If in addition* $G_2(x,t) < G_1(x,t)$ *for all* (x,t), *then* $y_2(t) < y_1(t)$ *when* $0 < t \leq T$.

Proof. On the interval $0 \leq t \leq T$ define $z(t) = y_1(t) - y_2(t)$,

$$\theta(t) = G_1(y_1(t),t) - G_2(y_1(t),t),$$

and

$$\varphi(t) = \begin{cases} \{G_2(y_2(t),t) - G_2(y_1(t),t)\}/z(t) & \text{if } z(t) \neq 0, \\ 0 & \text{if } z(t) = 0. \end{cases}$$

Clearly $\varphi(t)$ is nonnegative and measurable. Assumption (A6) implies $\varphi \in L^\infty(0,T)$. Using the definitions of z, φ and θ it follows that

$$z(t) = \int_0^t a(t-s)\theta(s)ds - \int_0^t a(t-s)\varphi(s)z(s)ds.$$

Lemma 3 implies $z(t) \geq 0$. This proves (i).

To prove part (ii) note that $\theta(s) > 0$ in this case. Therefore $z(t) > 0$ by Corollary 3. This proves Theorem 5.

LEMMA 6. *Suppose (A1-6) are satisfied. If G is independent of t and $u(t) \equiv M \geq 0$, then the solution $x(t)$ of (N) exists for all $t \geq 0$ and satisifes $0 \leq x(t) \leq M$.*

Proof. If $M = 0$, then $x \equiv 0$. If $M > 0$ let $y = M-x$ so that

$$y(t) = M-\int_0^t a(t-s)G(M-x(s))ds.$$

Theorem 1 above implies that $0 \leq y(t) \leq M$ for all $t \geq 0$. This proves Lemma 6.

THEOREM 6. *Suppose (A1-6) are satisfied. If m, $M \geq 0$ are such that $-m \leq u(t) \leq M$ on $0 \leq t \leq T$ and if $x(t)$ is the solution of (N), then $x(t)$ exists on $[0,T]$ and satisfies $-m \leq x(t) \leq M$. In particular $x(t)$ exists for all $t \geq 0$.*

Proof. Our proof is essentially the same as Padmavally's, c.f. [11, pp. 544-545].

Define a function H by

$$H(x) = \sup\{G(x,t);\ 0 \leq t \leq T\}.$$

Clearly $H(x)$ exists and is nonincreasing. Since

$$|H(x)-H(y)| = |\sup_t G(x,t)-\sup_t G(y,t)| \leq \sup_t |G(x,t)-G(y,t)|,$$

it follows that H is locally Lipschitz continuous.

Let us define that $N = \sup u(t)$ on the interval $0 \leq t \leq T$. Pick t_n such that $u(t_n) \to N$ and $0 \leq t_n \leq T$. Then

$$H(N) = \lim_n H(u(t_n)) = \lim_n\{\sup_t G(u(t_n),t)\} \leq \lim_n \sup\{G(u(t_n),t_n)\} = 0.$$

If $x < N$, then there exists a $t_o \in [0,T]$ such that $u(t_o) \geq x$. Therefore $H(x) = \sup_t G(x,t) \geq G(x,t_o) \geq 0$, and so $H(N) = 0$. This shows that the function $u^*(t)$ corresponding to H can be taken to be $u^*(t) \equiv N \geq 0$. If $N \geq 0$, then we may assume $M = N$. If $N < 0$, then we may assume $M = 0$. In this case replace $H(x)$ by $H(x+N)$.

Let $y(t)$ solve the equation

$$y(t) = \int_0^t a(t-s)H(y(s))ds.$$

By Lemma 6, $y(t)$ exists and $0 \leq y(t) \leq M$ for all $t \geq 0$. By Theorem 5, $M \geq y(t) \geq x(t)$ for as long as $x(t)$ exists and $0 \leq t \leq T$. The same argument shows that if $X = -x$, then $m \geq X(t) = -x(t)$ for as long as $x(t)$ exists and $0 \leq t \leq T$. Therefore $x(t)$ exists and satisfies $-m \leq x(t) \leq M$ on the interval $0 \leq t \leq T$. The last line of Theorem 6 follows from (A5) and the first part of the theorem. This proves Theorem 6.

THEOREM 7. *Theorems 4, 5 and 6 remain true if assumption (A6) is replaced by*

(A7) *$G(x,t)$ is uniformly continuous in x over each finite rectangle $x_1 \leqq x \leqq x_2$, $0 \leqq t \leqq t_1$.*

The proof of Theorem 7 is the same as the proof given by Padmavally in [11, pp. 547-549].

IV. MONOTONE SOLUTIONS

The purpose of this section is to generalize Theorems IV and V of [11] (compare [2, p. 391 and section 3]).

THEOREM 8. *Suppose (A1-5) and either (A6) or (A7) hold. If $G(x,t)$ is nondecreasing in t for each fixed x and $u(0) \geqq 0$, then the solution $x(t)$ of (N) is nonnegative and nondecreasing on $0 \leqq t < \infty$. If in addition (A6) holds and either*

(a) $u(0) > 0$ and $G(x,t) > 0$ when $x < u(t)$, or

(b) $G(x,t)$ is (strictly) increasing in t,

then $x(t)$ is increasing on $0 \leqq t < \infty$.

Proof. It is enough to consider the case where (A6) rather than (A7) is satisfied, c. f. [11, pp. 547-549]. Fix $h > 0$ and define

$$\varphi(t) = \{G(x(t),t) - G(x(t+h),t)\}/\{x(t+h)-x(t)\}$$

if $x(t+h) \neq x(t)$ and $\varphi(t) = 0$ otherwise. Let z_1 and z_2 solve the equations

$$z_1(t) = \int_0^h a(t+h-s)G(x(s),s)ds - \int_0^t a(t-s)\varphi(s)z_1(s)ds,$$

and

$$z_2(t) = \int_0^t a(t-s)\{G(x(s+h),s+h) - G(x(s+h),s)\}ds - \int_0^t a(t-s)\varphi(s)z_2(s)ds.$$

Since G is nondecreasing in t, the function $u(t)$ is nondecreasing. By Theorem 6 above $0 \leqq x(t) \leqq u(t)$ for $0 \leqq t < \infty$. Thus $G(x(t),t) \geqq 0$ for all $t \geqq 0$. Lemma 4 implies that $z_1(t) \geqq 0$ for all $t \geqq 0$. Since G is nondecreasing in t, $z_2(t) \geqq 0$ by Lemma 3 above. Therefore

$$x(t+h) - x(t) = z_1(t) + z_2(t) \geqq 0$$

for all $t \geqq 0$. Since $h > 0$ is arbitrary the function x is nondecreasing.

If (a) holds then by Corollary 5 $z_1(t) > 0$ for all $t > 0$. If (b) holds then by Corollary 4 $z_2(t) > 0$ for all $t > 0$. In either case $x(t+h) - x(t) > 0$ for all $t > 0$ and all $h > 0$. This proves Theorem 8.

THEOREM 9. *Suppose* (A1-5) *and either* (A6) *or* (A7) *are satisfied. Suppose* $G(x,t)$ *is nondecreasing in* t *for each fixed* x, $u(0) \geqq 0$ *and* $0 < u(\infty) < \infty$.

(i) *Suppose for each* $\delta > 0$ *there exists* $U > 0$ *and* $\varphi(\delta) > 0$

such that $G(x,t) \geq \varphi(\delta)$ *if* $t \geq U$ *and* $0 < x < u(t)-\delta$. *If* $a \notin L^1(0,\infty)$ *then the solution* $x(t)$ *of* (N) *tends to the limit* $u(\infty)$ *as* $t \to \infty$.

(ii) *If* $G(x+u(t),t) \to 0$ *as* $(x,t) \to (0^-,\infty)$ *and if* $a \in L^1(0,\infty)$ *then the solution* $x(t)$ *of* (N) *tends to a limit* $x(\infty) < u(\infty)$ *as* $t \to \infty$.

Proof. Note that $u(t)$ is bounded and nondecreasing so that $u(\infty)$ exists. Theorem 6 above implies that $0 \leq x(t) \leq u(t) \leq u(\infty)$ for all $t \geq 0$. Theorem 8 implies that $x(t)$ is nondecreasing. Therefore $x(\infty)$ exists and $0 \leq x(\infty) \leq u(\infty)$.

To prove part (i) suppose $u(\infty) - x(\infty) = \theta > 0$. Then there exists a $T > 0$ such that $G(x(t),t) \geq \varphi(\theta/2) > 0$ for all $t \geq T$. Therefore as $t \to \infty$

$$x(t) = \int_0^t a(t-s)G(x(s),s)ds \geq \int_0^T a(t-s)G(x(s),s)ds + \int_T^t a(t-s)\varphi(\theta/s)ds \to \infty.$$

This contradicts the boundedness of $x(t)$.

To prove part (ii) suppose $x(\infty) = u(\infty)$. Given $\varepsilon > 0$ there exist T_0 and $\varphi > 0$ such that if $t \geq T_0$ and $0 < u(t)-x < \varphi$ then $0 \leq G(x,t) < \varepsilon$. There exists $T \geq T_0$ such that $0 < u(t)-x(t) < \varphi$ for all $t \geq T$. Therefore when $t \geq T$

$$x(t) = \int_0^t a(t-s)G(x(s),s)ds \leq \int_0^T a(t-s)G(x(s),s)ds + \varepsilon\int_T^t a(t-s)ds$$

$$\to \varepsilon\int_0^\infty a(s)ds. \qquad (t \to \infty).$$

Since $\varepsilon > 0$ is arbitrary $0 = x(\infty) = u(\infty) > 0$. This contradiction proves part (ii). This completes the proof of Theorem 9.

V. PERIODIC CASE

Assume that G satisfies (A4) and the following additional assumptions.

(A8) $G(x,t)$ is locally Lipschitze continuous in x with Lipschitze constants independent of $t \in [0,\infty)$.

(A9) There is a function $u(t)$ and a constant $M > 0$ such that $G(u(t),t) = 0$ and $|u(t)| \leq M$ for all $t \geq 0$.

(A10) There exists a measurable function $G_o(x,t)$ defined for $-\infty < x$, $t < \infty$ and 2Π-periodic in t such that

$$\lim_{t \to \infty} |G(x,t)-G_o(x,t)| = 0$$

uniformly for x on compact subsets of $(-\infty,\infty)$.

LEMMA 7. If G satisfies (A4) and (A8-10) then

(i) $G_o(x,t)$ is nonincreasing in x for each fixed t,

(ii) G_o satisfies (A8)

(iii) $G(x,t)$ and $G_o(x,t)$ are bounded on sets of the form $|x| \leq K$, $0 \leq t < \infty$ for any fixed $K > 0$, and

(iv) there exists a bounded 2Π-periodic function $u_o(t)$ such that $G_o(u_o(t),t) = 0$ for all $t \geq 0$ and $|u_o(t)| \leq M$.

Proof. Only part (iv) needs comment. Fix any $t \in [0,2\Pi)$. Since the sequence $\{u(2n\Pi+t);\ n = 1, 2, 3,\ldots\}$ is bounded, there is a subsequence and a number $u_o(t)$ such that

$$u(2n_j\Pi+t) - u_o(t) \to 0. \qquad (j \to \infty)$$

Therefore if $m = 0, 1, 2,\ldots$

$$0 = \lim_{j\to\infty} G(u(2n_j\Pi+t), 2n_j\Pi+t) = G_o(u_o(t), 2m\Pi+t).$$

This defines $u_o(t)$ on $[0,2\Pi)$. Now extend $u_o(t)$ periodically. Note that there is no loss of generality in assuming either that 2Π is the least period of $G_o(x,t)$ in t or that G_o is independent of t. In either case $u_o(t)$ may be defined so that it has the same least period. This proves Lemma 7.

THEOREM 10. *If (A1-4) and (A8-10) are satisfied, then the solution $x(t)$ of (N) exists for all $t \geq 0$ and satisfies $|x(t)| \leq M$. If in addition either*

(a) *$G_o(x,t)$ is (strictly) decreasing in x for each fixed t, or*

(b) $a(t) \in L^1(0,\infty)$,

then there exists a 2Π-periodic, continuous function φ such that $x(t)-\varphi(t) \to 0$ as $t \to \infty$. The function φ has the same least period as G_o and is a constant if G_o is independent of t.

Proof. Theorem 6 and (A9) imply that $x(t)$ exists and $|x(t)| \leq M$ for all $t \geq 0$. Let L be the Lipschitze constant for G when $|x| \leq 2M+1$, $0 \leq t < \infty$. For any $N > L$, $x(t)$ satisfies the equation

$$x(t) = \int_0^t \{Na(t-s)\}\{G(x(s),s)/N\}ds.$$

The constant N can be picked so large that

$$(13) \qquad |G(x,t)/N - G(y,t)/N| \leq (1/2)|x-y|$$

when $|x|, |y| \leq 2M+1$. Also pick N so large that

$$(14) \qquad |G(x,t)/N| < M. \qquad (|x| \leq 2M+1,\ t \geq 0)$$

Since the functions $Na(t)$ and $G(x,t)/N$ also satisfy (A1-4) and (A8-10) it is no loss of generality to assume that (13) and (14) are true when $N = 1$.

Define a set of functions

$$S = \{z;\ z \text{ is continuous, } 2\Pi\text{-periodic and } |z(t)| \leq 2M \text{ for all } t \geq 0\}.$$

(If G_0 is independent of t, replace S by the subset of S consisting of constant functions.) For any $z \in S$ define

$$Tz(t) = \int_{-\infty}^{t} k(t-s)\{z(s)+G_o(z(s),s)\}ds, \qquad (-\infty < t < \infty)$$

where k is the solution of the resolvent equation (R). Clearly T is a completely continuous mapping of S into the set of all continuous 2Π-periodic functions with the uniform norm.

In order to show that $T: S \to S$ fix $z \in S$ and $t \in [0,2\Pi)$. If $M \leqq z(t) \leqq 2M$, then line (14) implies that $0 \leqq -G_o(z(t),t) \leqq M$. Therefore $0 \leqq z(t) + G_o(z(t),t) \leqq 2M$. Similarly if $-2M \leqq z(t) \leqq -M$, then $0 \geqq z(t) + G_o(z(t),t) \geqq -2M$. If $|z(t)| \leqq M$, then

$$|z(t) + G_o(z(t),t)| \leqq |z(t)| + |G_o(z(t),t)| \leqq 2M.$$

Since $t \in [0,2\Pi)$ is arbitrary it follows that

$$|Tz(t)| \leqq \int_{-\infty}^{t} k(t-s)2Mds = \{\int_{0}^{\infty} k(s)ds\}2M \leqq 2M.$$

Since $z \in S$ is arbitrary, $T: S \to S$.

By the Schauder Fix Point Theorem there exists a function $\varphi \in S$ such that

$$\text{(15)} \qquad \varphi(t) = \int_{-\infty}^{t} k(t-s)\{\varphi(s) + G_o(\varphi(s),s)\}ds. \qquad (-\infty < t < \infty)$$

This function φ is the unique 2Π-periodic continuous solution of (15). For if θ is another such solution distinct from φ, then there exists $t_o \in [0,2\Pi)$ such that

$$|\varphi(t_o)-\theta(t_o)| = \sup\{|\varphi(t)-\theta(t)| ; 0 \leq t < 2\Pi\} > 0.$$

Define $r(t) = \varphi(t+t_o) - \theta(t+t_o)$ and

$$g(t) = \{G_o(\theta(t+t_o),t+t_o) - G_o(\varphi(t+t_o),t+t_o)\}/r(t)$$

when $r(t) \neq 0$ and $g(t) = 0$ if $r(t) = 0$. Clearly $r(t)$ is measurable. By (A10) and line (13) it follows that $0 \leq g(t) \leq 1/2$ for all $t \geq 0$. Moreover $r(t)$ satisfies the equation

$$r(t) = \int_{-\infty}^{t} k(t-s)\{1-g(s)\}r(s)ds. \qquad (-\infty < t < \infty)$$

Suppose condition (a) is satisfied. Then there exists $t_1 > 0$ such that $k(t)$, $r(t)$ and $g(t)$ are all strictly positive on the interval $(0,t_1)$. Therefore we obtain the contradiction

$$|r(0)| < \int_{-\infty}^{0} k(-s)|r(s)|ds \leq \sup_t |r(t)| \int_0^{\infty} k(s)ds = |r(0)| \int_0^{\infty} k(s)ds \leq |r(0)|.$$

Suppose condition (b) is satisfied. Then

$$|r(0)| \leq \int_{-\infty}^{0} k(-s)|r(s)|ds \leq |r(0)| \int_0^{\infty} k(s)ds < |r(0)|.$$

Therefore in either case (a) or (b) the solution φ of (15) is unique.

In order to show that $x(t) - \varphi(t) \to 0$ as $t \to \infty$ write $x(t)$ in the form

$$x(t) = F(t) + \int_0^t k(t-s)\{x(s) + G_0(x(s),s)\}ds, \tag{16}$$

where

$$F(t) = \int_0^t k(t-s)\{G(x(s),s) - G_0(x(s),s)\}ds \to 0$$

as $t \to \infty$. Define $w(t) = x(t) - \varphi(t)$ for all $t \geq 0$. Let $w_0 = \lim\sup w(t)$ and $w_1 = \lim\inf w(t)$ as $t \to \infty$. Without loss of generality it may be assumed that $w_0 \geq |w_1|$ (otherwise replace x by $-x$ and φ by $-\varphi$).

Suppose for contradiction that $w_0 > 0$. Pick $\varepsilon > 0$ and a positive, increasing sequence $\{t_n\}$ such that $w(t_n) \to w_0$ and $w(t_n) \geq 2\varepsilon$. Since

$$x(t) = \int_0^t k(t-s)\{x(s) + G(x(s),s)\}ds$$

is the convolution of a function in $L^1(0,\infty)$ with a function in $L^\infty(0,\infty)$, $x(t)$ is uniformly continuous. Therefore there exists $T > 0$ such that $w(t+t_n) \geq \varepsilon$ when $|t| \leq T$ and $n = 1, 2, 3, \ldots$

Now apply Theorem 1 of [7] to equation (16). By this result there exists a subsequence of t_n (which we again index by n), a number $\tau \in [0,2\Pi)$ and a function $X(t)$ such that $t_n \to \tau \pmod{2\Pi}$,

$$x(t+t_n) - X(t) \to 0 \qquad (n \to \infty)$$

uniformly for t on compact subsets of $(-\infty,\infty)$ and

$$X(t) = \int_{-\infty}^{t} k(t-s)\{X(s) + G_o(X(s),s+\tau)\}ds. \qquad (-\infty < t < \infty)$$

Since φ is 2Π-periodic and $t_n \to \tau \pmod{2\Pi}$,

$$\varphi(t+t_n) \to \varphi(t+\tau) \qquad (n \to \infty)$$

uniformly in t. Therefore $X(t) - \varphi(t+\tau) \geq \mathcal{E}$ if $-T \leqq t \leqq T$.

Define $W(t) = X(t) - \varphi(t+\tau)$. Since $W(t) = \lim w(t+t_n)$ as $n \to \infty$, $|W(t)| \leqq w_o$ for all $t \in (-\infty,\infty)$. Moreover W satisfies the equation

$$W(t) = \int_{-\infty}^{t} k(t-s)\{1-g_o(s)\}W(s)ds, \qquad (-\infty < t < \infty)$$

where

$$g_o(s) = \{G_o(\varphi(s+\tau),s+\tau) - G_o(X(s),s+\tau)\}/W(s)$$

if $W(s) \neq 0$ and $g(s) = 0$ if $W(s) = 0$. Clearly $g_0(t)$ is measurable and $0 \leqq g_0(s) \leqq 1$ on the interval $-\infty < t < \infty$.

Suppose condition (a) is satisfied. Then $g_0(t) > 0$ on the interval $-T \leqq t \leqq T$. Therefore we obtain the contradiction

$$w_0 = W(0) = \int_{-\infty}^{0} k(-s)\{1-g_0(s)\}W(s)ds < \int_{-\infty}^{0} k(-s)|W(s)|ds \leqq w_0 \int_0^\infty k(s)ds \leqq w_0.$$

If condition (b) is satisfied, the contradiction is

$$w_0 = W(0) \leqq \int_{-\infty}^{0} k(-s)|W(s)|ds \leqq \int_0^\infty k(s)w_0 ds < w_0.$$

Therefore one must have $w_0 = 0$. This shows that $x(t) - \varphi(t) \to 0$ as $t \to \infty$ and completes the proof of Theorem 10.

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